Stochastic Models for Portfolio Management with Minimum Transaction Lots

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Abstract
In this paper, we consider the problem of a decision maker who is concerned with the management of a portfolio over a finite horizon. The portfolio optimization problem involves portfolio rebalancing decisions in response to new information on market future prices of the risky assets. Rebalancing decisions are manifested in the revision of holdings through sales and purchases of assets. We assume that the assets are sufficiently liquid that market impacts can be neglected. Usually, portfolio selection problems are solved with quadratic or linear programming models. In the real life applications, each asset usually has its minimum transaction lot. Methods considering minimum transaction lots were developed based on some linear portfolio optimization models. In this paper, we study the minimum transaction lot problem in portfolio optimization based on Markowitz’ model, which is probably the most well-known and widely used. Based on Markowitz’ model, this study presents three possible models for portfolio selection problems with minimum transaction lots and discuss a two-parameter penalty algorithm to obtain the optimal solutions (decisions) for the portfolio problems.

Keywords: portfolio management, transaction lots, two-parameter penalty algorithm

JEL Classification: C02, C61, G11

Modele Stocastice pentru Managementul Portofoliilor cu Loturi Minime de Tranzacționare

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Rezumat
În această lucrare, ne ocupăm de problema managementului unui portofoliu de acțiuni considerat pe o perioadă finită de timp. Problema optimizării portofoliului implică luarea de decizii relativ la compoziția acestuia ca urmare a informațiilor noi relativ la prețurile viitoare ale acțiunilor. Deciziile implică revizuirea numărului de acțiuni devenite prin achiziționarea respectiv, vânzarea unora dintre acestea. Vom presupune că piața acțiunilor din portofoliu este suficient de lichidă astfel încât, dacă se decide vânzarea unui anumit număr de acțiuni, acestea să fie imediat absorbite de către piață. De obicei, problema optimizării portofoliului este rezolvată cu ajutorul modelelor pătratice sau liniare. În această lucrare, considerăm problema managementului unui portofoliu în care acțiunile sunt tranzacționate în loturi minime. Sunt propuse trei modele care au la bază modelul lui Markowitz și este discutat un algoritm bazat pe o funcție de penalitate cu doi parametri pentru găsirea soluțiilor (deciziilor) optime pentru problema managementului portofoliilor.

Cuvinte cheie: managementul portofoliilor, loturi de tranzacționare, algoritm de penalitate cu doi parametri

Clasificare JEL: C02, C61, G11

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1. INTRODUCTION

The main feature of the investment and financial problems is the necessity to make decisions under uncertainty over a time period. The uncertainties concern the future level of interest rates, yields of stock, exchange rates, prepayments, external cashflows, inflation, future demand, liabilities, etc. There exist various stochastic models describing or explaining these parameters. Numerous early studies dealt with the properties of the portfolio optimization problems based on different criteria; we refer to the collection Ziemba and Vickson [19] of original papers with commentaries, to the relatively recent survey Constantinides and Malliaris [20] and to textbook Elton and Gruber [21]. The well-known Markowitz model [1] became a standard tool for portfolio optimization. It has been applied not only to portfolios of shares, but also to bonds Mulvey and Zenios [22], to international loans Seppala [23], even to asset and liability management with portfolio returns replaced by the surplus Mulvey [24]. It was the introduction of risk into the investment decisions which was the exceptional feature of this model and a real breakthrough. Markowitz’ model [1] uses the mean and variance of historical returns to measure the expected return and risk of a portfolio. Such portfolio selection problems are solved with quadratic or linear programming models. In real life applications, each asset has its minimum transaction lot, while the solutions involve only real-number asset weights rather than asset trading units. For example, stocks might be traded at the unit one share, and mutual funds have their individual minimum trading amounts. Thus, the solution obtained by Markowitz’ model must be integers to be applicable in practice. Other than Markowitz’ model, Speranza [2], Mansini and Speranza [3,4], and Kellerer et al. [5] proposed their respective portfolio selection models based on Konno and Yamazaki’s mean absolute deviation (MAD) model [6]. Speranza [2] proposed a mixed integer program considering realistic characteristics in portfolio selection, such as minimum transaction lots and the maximum number of securities, and suggested a simple two-phase heuristic algorithm to solve the proposed integer program. Mansini and Speranza [4] showed that the portfolio selection problem with minimum transaction lots is an NP-complete problem and proposed three heuristic algorithms to solve the problem. Based on the MAD model, Konno and Wijayanayake [7] proposed an exact algorithm for portfolio optimization problems under concave transaction costs and minimum transaction lots. However, minimum transaction lots were not the major concern in their study. Later, Mansini and Speranza [8] derived a meansafety model with side constraints from the MAD model, and proposed an exact algorithm to solve for portfolios under the consideration of transaction costs and minimum transaction lots. However, Markowitz’ model is still the most widespread portfolio selection model. Solving the portfolio selection problem based on Markowitz’ model and, simultaneously, considering minimum transaction lots are of practical significance. However, it appears that no methods in the past solving the portfolio selection problem with minimum transaction lots were based on Markowitz’ model. Apart from considering minimum transaction lots, Markowitz’ model is intrinsically a multiobjective decision-making (MODM) problem whose decision criteria conflict with each other. The return is required to be maximized and the risk minimized. However, the risk is often high when the return is maximized, and the return is often low when the risk is minimized. Researchers have proposed different approaches, such as goal programming [10] and multiple objective programming [11], to solve multi-objective portfolio selection problems. Lee and Lerro [10] pioneered goal programming in portfolio selection, but their method is not based on Markowitz’ model. Arenas Parra et al. [12] proposed a fuzzy goal programming approach to solve a portfolio selection problem, using a multi-index model to estimate the return and risk of portfolios and treating fuzzy goals as fuzzy numbers. However, their method ignores the existence of minimum transaction lots. Apart from goal programming, fuzzy programming
can also be used to solve MODM problems [13], but conventional fuzzy programming methods cannot incorporate objective weights that are usually important for the decision maker (DM) to express his/her preference regarding return and risk. Lin [14] recently proposed a weighted max–min model to incorporate objective weights with fuzzy programming. Promisingly, this method can be applied to solving the portfolio selection problem. Using this method, the DM obtains portfolios through setting appropriate objective weights. The portfolio selection problem with minimum transaction lots is a combinatorial optimization problem whose feasible region is not continuous. Studies have shown that genetic algorithm (GA) is a promising approach to combinatorial optimization problems. Shoaf and Foster [9] applied GA to Markowitz’ portfolio selection problem and found that the time required by GA is better than that of quadratic programming. However, their approach does not consider the constraints of minimum transaction lots. The solution obtained will be much more realistic if the constraints of minimum transaction lots are considered when solving with GAs. As pointed out, many methods were applied to solve the great variety of models. In recent years, we can find a rich literature regarding the portfolio optimization problem, see for example Fulga [16-18] and the references therein.

The rest of this paper is organized as follows. Based on Markowitz’ model, Section 2 derives three models for portfolio selection with minimum transaction lots. Section 3 presents the two-parameter penalty algorithm proposed for solving the portfolio management problem. Conclusions and directions for future work are stated in Section 4.

2. PORTFOLIO MANAGEMENT MODELS

Originally an MODM problem, portfolio selection attempts to maximize the rate of return and minimize the portfolio risk simultaneously. In practice, the MODM problem is often degenerated to a single objective one by introducing a preference structure to compromise between the objectives, or simply optimizing one of the two objectives while bounding the remaining one. The latter approach leads to the well-known Markowitz’ model [1]. Markowitz’ model assumes that rational investors aim to maximize return under a certain risk level or minimize risk above a certain return level. For convenience, the decision maker usually fixes the expected rate of return and then minimizes the portfolio risk under this return constraint. Consequently, Markowitz’ model can be expressed as a quadratic programming problem as follows:

\[
\begin{align*}
&\min \sum_{i=1}^{n} \sum_{j=1}^{n} x_i x_j \sigma_{ij} \\
&\text{s.t. } \sum_{j=1}^{n} x_j r_j = r, \\
&\sum_{i=1}^{n} x_i = 1, \quad x_i \geq 0, \quad i = 1, n,
\end{align*}
\]

where \( n \) is the number of assets, \( x_i \) represents the weight of budget invested in asset \( i \), \( \sigma_{ij} \) is the return covariance between assets \( i \) and \( j \), \( r_i \) is the expected rate of return of asset \( i \), \( r \) is the required rate of return. We also use the notations \( \sigma^2_p = \sum_{i=1}^{n} \sum_{j=1}^{n} x_i x_j \sigma_{ij} \) for the return variance of the portfolio and \( ERR = \sum_{i=1}^{n} x_i r_i \) for the expected rate of return. As in the original Markowitz’ model, \( x_i \geq 0, \quad i = 1, n \) means that no short selling is allowed. However, in practice, each asset has its minimum transaction lot that should be taken into consideration in finding minimum-risk portfolios. Therefore, the portfolio selection models need to be modified to consider the minimum transaction lots. A modified Markowitz’ model can be formulated as follows to consider the minimum transaction lots:

\[
\begin{align*}
&\min \sum_{i=1}^{n} \sum_{j=1}^{n} x_i x_j \sigma_{ij} \\
&\text{s.t. } \sum_{j=1}^{n} x_j r_j = r, \\
&\sum_{i=1}^{n} x_i = 1, \quad x_i \geq 0, \quad i = 1, n,
\end{align*}
\]
where $\beta$ represents the budget, $p_i$ denotes the unit price of asset $i$ and $y_i$ represents the units invested in asset $i$. We denote the expected portfolio return by $EPR = \sum_{i=1}^{n} p_i y_i r_i$. Since the same rate of return can be achieved with different budgets, the first constraint of this model, $EPR \geq \beta r$, uses portfolio return instead of rate of return to maximally utilize the budget. More return is earned as more capital is invested. We require also that the total investment is below the budget and the sum of weights is equal to unity. Notably, this constraint causes the model to be nonlinear.

Similar to $(PP_1)$, $(PP_2)$ minimizes the investment risk with respect to a given rate of return $r$ except that the optimal solution must be in integers. However, the modification might make a portfolio with a rate of return equals $r$ unobtainable, namely, the rate of return might not exactly equal $r$. Owing to constraint regarding the expected portfolio return, the rate of return can only exceed $r$. In a portfolio selection problem whose solution is in real numbers, the desired rate of return along with its corresponding risk represents a target portfolio to be achieved. The target portfolio can always be achieved in a real-number solution space. However, in an integer solution space, the target portfolio might be unreachable, and the optimal solution can be far from the target. Specifically, the risk of the optimal solution to $(PP_2)$ might be too high to be acceptable if its rate of return is to exceed the desired $r$. Conversely, a portfolio with a rate of return slightly less than the desired $r$ might be more appealing if the portfolio risk can accordingly be lowered significantly. This alternative leads to another decision model that minimizes the distance between the obtained and the target portfolios. The model is as follows:

$$
\begin{align*}
\min & \left( \frac{\sigma - \sigma^*}{\sigma^* - \sigma}\right)^2 + \left( \frac{EPR - \beta r}{\beta r^* - \beta r}\right)^2 \\
\text{s.t.} & \quad \sigma^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} x_i x_j \sigma_{ij}, \\
& \quad EPR = \sum_{i=1}^{n} p_i y_i r_i, \\
& \quad \sum_{i=1}^{n} p_i y_i \leq \beta, \\
& \quad x_i = \frac{p_i y_i}{\sum_{i=1}^{n} p_i y_i}, \quad y_i \in \mathbb{Z}, \quad i = 1, n,
\end{align*}
$$

where $\sigma$ is the minimum risk corresponding to the desired $r$, $r^*$ and $\sigma^*$ represents the maximal rate of return respectively, the minimal risk that can be achieved, and $\sigma^*$ and $r^*$ are their corresponding risk and rate of return. Notably, the risk and return are normalized to eliminate the influence of scales. Therefore, $(\sigma^*, r^*)$ and $(\sigma^*, r^*)$ represent the lower and upper ends of the efficient frontier. Obviously, this model is also a nonlinear integer
programming problem that is difficult and time-consuming to solve. Next, we give an algorithm based on a two-parameter penalty function which can provide a suboptimal solution because of the lack of the integrality condition.

3. THE TWO-PARAMETER PENALTY ALGORITHM

We make the assumption that, at each local minimizer of the nonlinear programming problem \((PP_3)\), an appropriate constraint qualification is assumed to hold, thereby ensuring that any optimal point \(x^*\) of the nonlinear programming problem \((PP_3)\) satisfies the Karush-Kuhn-Tucker conditions regarding the existence of a vector of Lagrange multipliers \(\lambda = (\lambda_1^*, \lambda_2^*) \in R^m \times R^m\), which satisfies the necessary conditions. For the simplicity of the notations, the constraints of the model are considered as double inequalities as in \(a_i \leq c_i(x) \leq b_i, \ i = 1, m\).

3.1. The Penalty Function Problem

The nonlinear programming problem \((PP_3)\) is not solved directly; instead a non-differentiable exact penalty function \(\Phi\) is minimized, where the exact penalty function is constructed so that local minimizers of the nonlinear programming problem are also local minimizers of the penalty function \(\Phi\). The penalty function is \(\Phi = f + \mu \cdot \theta + \frac{1}{2} \nu \cdot \theta^2\), with \(\mu > 0, \nu \geq 0\) where \(\theta\) represents the degree of infeasibility, for details, see Fulga [15]. The penalty function \(\Phi\) may be viewed as a hybrid of a quadratic penalty function based on the infinity norm and a single parameter exact penalty function. Clearly \(\theta\) is continuous \(\forall x \in R^n\), but it is usually not differentiable for some \(x\). However, the directional derivative \(D_p \theta(x)\) exists for any \(x, p \in R^n\), \(D_p \theta(x) = \lim_{\alpha \to 0} \frac{\theta(x + \alpha \cdot p) - \theta(x)}{\alpha}\). We recall that, for fixed values of \(\mu > 0\) and \(\nu \geq 0\), a point \(x^*\) is a critical point of \(\Phi\) if and only if for all \(p \in R^n\) the directional derivative \(D_p \Phi(x^*)\) is non-negative and also, the solution set of the penalty function problem with fixed values for \(\mu > 0, \nu \geq 0\) is defined as the set of critical points of \(\Phi\). Due to Theorem 2.3. from Fulga [15], under some mild conditions, we have the equivalence between the optimal solutions of the nonlinear programming problem \((PP_3)\) and the critical points of \(\Phi\). Next, we give the outline of the two-parameter penalty algorithm.

3.2. The Algorithm

For purposes of ensuring convergence, the following bound is imposed at each iteration:
\[
\|p^k\|_\infty \leq S_{\text{bound}}.
\]

Step 1. Initialization
\[k = 1, \ \mu^1 = 1, \nu^1 = 1, \ H^1 = I, \ \epsilon = 10^{-5}, \ \rho = 0.02, \ \delta = 10^{-8}, \ S_{\text{bound}} = 10^{10}, \ \theta_{\text{cross}} = 1, \ \theta_{\text{cap}} = 100, \ k_1 = 1.2, \ k_2 = 1.5, \ k_3 = 1.2, \ k_4 = 4.\]

Step 2. Update \(H\) and the penalty parameters
\(H\) is updated using The Broyden-Fletcher-Goldfarb-Shanno update provided this maintains positive definiteness (this step is omitted for the first iteration); otherwise \(H\) is not updated.

(i) If \(\theta^k < \theta_{\text{cross}}\) and \(\mu^k < k_1 \| \lambda^k \|\) then \(\mu^{k+1} = k_2 \| \lambda^k \|, \ \nu^{k+1} = \nu^k\).

(ii) If \(\theta^k > \theta_{\text{cross}}\) and \(\mu^k + \nu^k \theta^k < k_4 \| \lambda^k \|\) then \(\mu^{k+1} = \mu^k, \ \nu^{k+1} = \frac{k_4 \| \lambda^k \| - \mu^k}{\theta^k}\).
Step 3. Solve the \((P^k)\) problem

\[
\begin{align*}
\min_{p, x} & \quad p^T \nabla f(x^k) + \frac{1}{2} \| p - H^k \cdot p + \mu^k \cdot \xi + \frac{1}{2} v^k \cdot \xi^2 \\
\text{subject to} & \quad c_i(x^k) - b_i + p^T \nabla c_i(x^k) \leq \zeta, \quad i = 1, m \\
& \quad a_i - c_i(x^k) + p^T (\nabla c_i(x^k)) \leq \zeta, \quad i = 1, m \\
& \quad \zeta \geq 0.
\end{align*}
\]

If \( \theta^k > \theta_{\text{cap}} \) then the capping constraint \( \zeta \leq \theta^k \) is also imposed. Then this problem is solved. If the capping constraint is not active at the \((P^k)\)’s solution, then the algorithm proceeds directly to Step 4. Otherwise, the penalty parameters are updated as described in Step 2, except that \( \|\lambda^k\| \) is replaced by \( \mu^k + v^k \theta^k + |\xi| \), where \( \xi \) is the Lagrange multiplier of the capping constraint. The \((P^k)\) problem is then solved again.

Step 4. Attempt the proposed step

If (i) \( \Phi(x^k) - \Phi(x^k + p^k) \geq p[\Psi^k(0) - \Psi^k(p^k)] \)

(ii) either the penalty parameters were not altered in Step 3 or the inequality \( \theta(x^k + p^k) \leq \theta(x^k) \) is satisfied,

then the proposed step is accepted and the algorithm proceeds to step 7.

Step 5. Calculate the Maratos effect correction vector

Solve the following quadratic problem for the second order correction \( t^k \):

\[
\begin{align*}
\min_{t \in R^k} & \quad \| t \|_2^2 \\
\text{subject to} & \quad c_i(x^k + p^k) - b_i + t^T \nabla c_i(x^k) \geq 0, \quad a_i - c_i(x^k + p^k) - t^T \nabla c_i(x^k) \geq 0, \quad \forall i \in T
\end{align*}
\]

where \( T \) is the set of indices of the constraints active at the \((P^k)\)’s solution in Step 3. If \( \| t^k \|_2 \geq \| p^k \|_2 \) then set \( t^k = 0 \).

Step 6. Arc search

Consider successive values of the sequence \( 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots \) as trial values of \( \alpha \). If \( t^k = 0 \), then omit the first member of the sequence. Accept the first trial value which satisfies

(i) \( \Phi(x^k) - \Phi(x^k + q^k(\alpha)) \geq p\alpha[\Psi^k(0) - \Psi^k(p^k)] \) where \( q^k(\alpha) = \alpha p^k + \alpha^2 t^k \)

(ii) If the penalty parameters were altered in Step 3, then the step \( q^k(\alpha) \) is also required to satisfy the condition \( \theta(x^k + q^k(\alpha)) \leq \theta_k \). After a satisfactory value of \( \alpha \) has been found, set \( x^{k+1} = x^k + q^k(\alpha) \).

Step 7. Check the stopping conditions

The algorithm halts if either the length of the previous step \( \| x^k - x^{k-1} \|_2 \leq \delta \) or

(i) \( \theta^k < \epsilon \) and

(ii) \( \| \nabla f(x^k) + \sum_{i \in A} \lambda^k_i \nabla c_i(x^k) - \sum_{j \in B} \lambda^k_j \nabla c_j(x^k) \|_2 < \epsilon \)

where \( A^k = \{ i | c_i(x^k) - b_i < 10^{-5} \} \), \( B^k = \{ j | a_j - c_j(x^k) < \epsilon \} \). Otherwise, \( k \) is incremented, and the algorithm proceeds to Step 2.

4. CONCLUSIONS

This study proposes decision models for portfolio selection problems with minimum transaction lots and uses a two-parameter penalty algorithm to solve the models. The proposed models include one model directly modified from the well-known Markowitz’ model and one other model minimizing the distance between the target and obtained
portfolios. The proposed algorithm generates convergent sequences under mild conditions; it is effective in practice and the use of the second penalty parameter significantly reduces the effort required to solve constraint nonlinear programs. The two-parameter exact penalty function based on the infinity norm of constraint violations has an advantage over one-norm based exact penalty function in that only the gradients of the most violated constraints need be calculated in order to find a search direction; for one-norm exact penalty functions, the gradients of all active and violated constraints may be required. As a future direction of research, we will consider the portfolio optimization models over more than one time period.

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